Trends in *p*-adic Function Theory

William Cherry
University of North Texas

Dalat University
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Definition (p-adic absolute value $|\ |_p$)

Let p be a prime number, let x be a rational number, and define

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-n} & \text{if } x = p^n \frac{a}{b}, \text{where } a \text{ and } b \text{ are integers with } p \nmid ab. \end{cases}$$

Examples

- $|12|_2 = \dfrac{1}{4}$;
- $\left|\dfrac{7}{375}\right|_7 = \dfrac{1}{7}$;
- $\left|\dfrac{7}{375}\right|_5 = 125$;
- $\left|\dfrac{7}{375}\right|_{11} = 1$.

Proposition

The p-adic absolute value $|\ |_p$ satisfies the following properties:

AV 1 *$|x|_p$ is a non-negative real number and $|x|_p = 0$ if and only if $x = 0$;*

AV 2 $|xy|_p = |x|_p \cdot |y|_p$.

AV 3 $|x+y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$ *(non-Archimedean)*

p-adic numbers

- The completion of the rational numbers $\mathbf{Q}$ with respect to the p-adic absolute value is called the field of p-adic numbers and is denoted $\mathbf{Q}_p$.
  - The field $\mathbf{Q}_p$ is a complete locally compact field, somewhat analogous to the field of real numbers.
  - The topology on $\mathbf{Q}_p$ induced by $|\ |_p$ is totally disconnected.
  - The p-adic numbers were created by Hensel. **Hensel's lemma** extend's Newton's method to the p-adic numbers and transforms the question of whether a polynomial with integer coefficients has a zero in the p-adic numbers into a question about whether the polynomial has a zero modulo a large enough power of p.
  - A classical question in number theory is when does a quadratic form in several variables and with integer coefficients have a non-trivial rational zero? For instance, are there rational numbers (x, y, z) such that

    $$x^2 + y^2 + 13xy - 107xz + 46yz = 0?$$

    A theorem of **Minkowski** says that a quadratic form with rational coefficients has a non-trivial rational zero if and only if it has a non-trival real zero and also a non-trivial p-adic zero for all p. This was extended to any number field by **Hasse**, and since then the p-adic numbers have been of great interest to number theorists. See Serre's *A Course in Arithmetic* for a nice introduction to this subject.

p-adic complex numbers

- Let $\mathbf{C}_p$ denote the completion of the algebraic closure of $\mathbf{Q}_p$. The field $\mathbf{C}_p$ is called the *p*-**adic complex numbers.**

Proposition

The field $\mathbf{C}_p$ is algebraically closed.

Remark

The field $\mathbf{C}_p$ is **not** *locally compact, and therefore there is no Ascoli-Arzela type theorem for functions on $\mathbf{C}_p$.*

Residue Class Field

- By multiplicativity and the non-Archimedean triangle inequality, the set

$$\mathcal{O}_{\mathbf{C}_p} = \{z \in \mathbf{C}_p : |z|_p \leq 1\}$$

  forms a subring of $\mathbf{C}_p$, called the **ring of integers** of $\mathbf{C}_p$.
- Similarly, one sees that the set

$$\mathfrak{m}_{\mathbf{C}_p} = \{z \in \mathbf{C}_p : |z|_p < 1\}$$

  forms a maximal ideal in $\mathcal{O}_{\mathbf{C}_p}$, and in fact the unique maximal ideal in $\mathcal{O}_{\mathbf{C}_p}$.
- The field $\mathcal{O}_{\mathbf{C}_p}/\mathfrak{m}_{\mathbf{C}_p}$ is called the **residue class field,** and in this case it is easy to see that it is isomorphic to $\mathbf{F}_p^a$, the algebraic closure of the finite field of p elements.
- If z is in $\mathcal{O}_{\mathbf{C}_p}$, then we denote by $\tilde{z}$ the image of z in the residue class field $\mathbf{F}_p^a$.

Remark

- *It is not so important to work with exactly the field* $\mathbf{C}_p$.
- *This is just one of many complete algebraically closed non-Archimedean fields.*
- *I am using* $\mathbf{C}_p$ *in my lecture today to give a concrete example.*
- *Much of what I am discussing today remains true for any non-Archimedean field which is complete and algebraically closed.*
- *In some cases it is also important that* $\mathbf{C}_p$ *has characteristic zero, i.e., contains the rational numbers* $\mathbf{Q}$.
- *Note, however, that* $\widetilde{\mathbf{C}_p}$ *has positive characteristic* p, *and so the residue class field need not have characteristic zero.*

Remark

A series $\sum_{k=0}^{\infty} a_k$ *converges in* $\mathbf{C}_p$ *if and only if* $\lim_{k\to\infty} |a_k|_p = 0$.

Entire and Meromorphic Functions

Definition

By an **entire** function f on $\mathbf{C}_p$, we mean simply a formal power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

with coefficients a_k in $\mathbf{C}_p$ and with infinite radius of convergence.
A **meromorphic** function f is the quotient $f = f_1/f_0$ of two entire functions f_0 and f_1 with the denominator f_0 not the zero function.
A Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

is said to be **analytic** on $r_1 < |z| \leq r_2$ if it converges for all such z.

Maximum Modulus Principle

Let $f(z) = \sum a_k z^k$ be an entire function and let $r > 0$ be a positive real number.

Definition

$$|f|_r = \max_k |a_k|_p r^k.$$

If $|c|_p \leq r$, then the non-Archimedean triangle inequality immediately implies

$$|f(c)|_p \leq \max_k |a_k|_p |c|_p^k \leq \max_k |a_k|_p r^k = |f|_r.$$

Proposition (Maximum Modulus)

Let $r = |b| > 0$. There exists a non-zero polynomial $P(z)$ in $\mathbf{F}_p^a[z]$ such that for all z in $\mathbf{C}_p$ with $|z|_p \leq r$, we have

$$|f(z)|_p = |f|_r,$$

unless $\widetilde{\left(\frac{z}{b}\right)}$ is a root of P.

Proof of Maximum Modulus Principle

Proof.

- Choose c such that $|c|_p = \max_k |a_k b^k|_p = \max_k |a_k|_p r^k = |f|_r$.
- Suppose $|z|_p \leq r$ and that $|f(z)|_p < |f|_r = |c|_p$.
- This precisely means $\left| \sum_k \frac{a_k}{c} z^k \right|_p < 1$.
- Hence, $\left| \sum_k \frac{a_k}{c} b^k \left(\frac{z}{b}\right)^k \right|_p < 1$.
- Since $a_k b^k / c$ are elements of $\mathcal{O}_{\mathbf{C}_p}$, this exactly means $\sum_k \widetilde{\left(\frac{a_k b^k}{c}\right)} \widetilde{\left(\frac{z}{b}\right)} = 0$, and

  at least one but only finitely many $\widetilde{\left(\frac{a_k b^k}{c}\right)} \neq 0$.

□

A consequence of the Maximum Modulus Principle

Corollary

If g and h are entire and $r > 0$, then

$$|gh|_r = |g|_r |h|_r.$$

Proof.

If there exists $b \in \mathbf{C}_p$ with $|b|_p = r$, then by the Maximum Modulus Principle, there exists c in $\mathbf{C}_p$ such that

$$|gh|_r = |g(c)h(c)|_p = |g(c)|_p |h(c)|_p = |g|_r |h|_r.$$

□

We can therefore extend $| \ |_r$ to meromorphic functions $f = f_1/f_0$ by

$$|f|_r = \frac{|f_1|_r}{|f_0|_r}$$

The Valuation Polygon

Again, let $f(z) = \sum a_k z^k$. If there exists k_0 such that $|a_k|_p r^k < |a_{k_0}|_p r^{k_0}$ for all $k \neq k_0$, and if $|z| = r$, then by the non-Archimedean triangle inequality

$$\left| \sum_{k \neq k_0} a_k z^k \right|_p < \left| a_{k_0} z^{k_0} \right|_p ,$$

and so, again by the non-Archimedean triangle inequality, $|f(z)|_p = |f|_r$. Thus, if $f(z) = 0$, then for $r = |z|$, the number of indices k such that $|a_k|_p r^k = |f|_r$ is at least two. Such values of r are called **critical points**.

The Valuation Polygon

Proposition (Theory of Newton or Valuation Polygons)

Let $f(z) = \sum_k a_k z^k$ *be an entire function, and let*

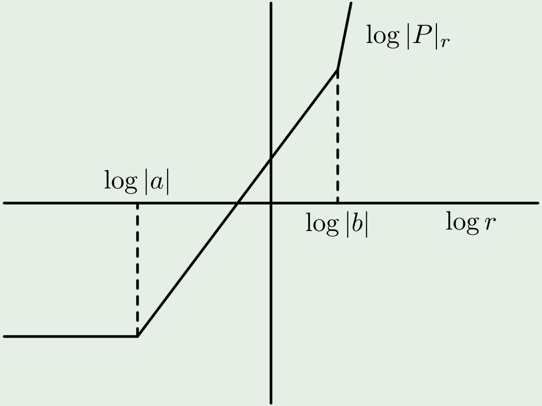
$$k(f,r) = k(r) = \min\{k : |a_k|_p r^k = |f|_r\} \text{ and } K(f,r) = K(r) = \max\{k : |a_k|_p r^k = |f|_r\}.$$

Then, f *has precisely* $K(r) - k(r)$ *zeros, counting multiplicity, with modulus* r. *Moreover, the set of* r *such that* $K(r) > k(r)$ *is discrete.*

The Valuation Polygon

An Example of a Valuation Polygon

Let $P(z) = (z-a)^n(z-b)^m$ with $|a|_p < |b|_p$. The following is the graph of $\log |P|_r$ as a function of $\log r$.



Consequences of Valuation Polygons: Picard Theorems

Corollary (Picard Theorems)

- *A non-constant entire function takes on* **every** *value in* $\mathbf{C}_p$.
- *A transcendental entire function takes on every value in* $\mathbf{C}_p$ *infinitely often.*
- *A function analytic in* $0 < |z| \leq 1$ *with an essential singularity at the origin takes on every value in* $\mathbf{C}_p$ *infinitely often.*

Proof Sketch.

Let c be in $\mathbf{C}_p$ and consider the series expansion for $f(z) - c$. Take $r > 0$ and suppose $K(f - c, r) = k(f - c, r)$; call this k. Let j be an adjacent index such that $a_j \neq 0$. Solving

$$|a_k|_p r^k = |a_j|_p r^j$$

locates an r where $K(r) > k(r)$, and hence a zero of $f(z) - c$. □

Consequences of Valuation Polygons: Riemann Mapping Theorem

Corollary (Riemann Mapping)

The image of a disc under an entire (or analytic) map is again a disc.

Proof Sketch.

- Let f be a non-constant analytic function in $|z| \leq 1$.
- Suppose $f(0) = 0$.
- Let c be an element of $\mathbf{C}_p$.
  - If $|c|_p > |f|_1$, then $k(f - c, r) = K(f - c, r) = 0$ for all $r \leq 1$, so c is not in the image of f.
  - If $0 < |c|_p \leq |f|_1$,
    - for r close to zero, $|f - c|_r = |c|_p$ and $k(f - c, r) = K(f - c, r) = 0$.
    - For $r = 1$, we have $|f - c|_1 = |f|_1 \geq |c|_p$.

  Hence $K(f - c, 1) > 0$, and so for some $r \leq 1$, we have $k(r) < K(r)$. □

Consequences of Valuation Polygon: Jensen Formula

Definition

$$N(f,0,r) = \mathrm{ord}_0^+ f \log r + \sum_{z \neq 0} \mathrm{ord}_z^+ f \log \frac{r}{|z|_p}$$

Proposition (Jensen Formula)

If f is a non-constant entire function, then as $r \to \infty$,

$$N(f,0,r) = \log |f|_r + O(1).$$

Logarithmic Derivative Lemma

Proposition (Logarithmic Derivative Lemma)

If f is meromorphic, $\left|\frac{f'}{f}\right|_r \leq \frac{1}{r}$.

Proof Sketch.

In the entire case, write $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

- $f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$.
- $|k a_k|_p r^{k-1} \leq \frac{|a_k|_p r^k}{r}$ since $|k|_p \leq 1$.
- $\Rightarrow |f'|_r \leq \frac{|f|_r}{r}$.

□

Logarithmic Derivative Lemma

Proposition (Logarithmic Derivative Lemma)

If f is meromorphic, $\left|\dfrac{f'}{f}\right|_r \leq \dfrac{1}{r}$.

Corollary (Picard's Theorem)

If an entire function f is zero-free, then f is constant.

Proof.

If f is zero-free, then f'/f is entire. But,

$$\left|\frac{f'}{f}\right|_r \leq \frac{1}{r} \to 0 \text{ as } r \to \infty,$$

and so $f'/f \equiv 0$, meaning f must be constant. □

Value Sharing

Theorem (Value Sharing [Adams & Straus])

Let $f = f_1/f_0$ and $g = g_1/g_0$ be a meromorphic function and let $c_1, \ldots, c_4$ be four values in $\mathbf{C}_p$. If $f^{-1}(c_j) = g^{-1}(c_j)$ for $j = 1, \ldots, 4$, then either both f and g are constant or $f = g$.

Proof Sketch.

- Assume $\max\{|g_0|_r, |g_1|_r\} \leq \max\{|f_0|_r, |f_1|_r\}$ for a sequence of $r \to \infty$.
- Consider the **entire** function $h = \dfrac{(f_1 g_0 - f_0 g_1)(f_0 f_1' - f_1 f_0')}{(f_1 - c_1 f_0)(f_1 - c_2 f_0)(f_1 - c_3 f_0)(f_1 - c_4 f_0)}$.
- $|f_1 - c_j f_0|_r \geq \max\{1, |c_j|_p\} \max\{|f_0|_r, |f_1|_r\}$ for r large.
- $|f_1 g_0 - f_0 g_1|_r \leq (\max\{|f_0|_r, |f_1|_r\})^2$.
- $|f_0 f_1' - f_1 f_0'|_r = |f_0 f_1|_r \left| \dfrac{f_1'}{f_1} - \dfrac{f_0'}{f_0} \right|_r \leq (\max\{|f_0|_r, |f_1|_r\})^2 \dfrac{1}{r}$.
- $\Rightarrow |h|_r \leq \dfrac{\mathrm{const}}{r}$ for a sequence of $r \to \infty$. □

Maps to Varieties

Definition

A **variety** *X is the set of solutions of a finte system of polynomial equations*

$$P_j(X_0, \ldots, X_n) = 0.$$

- If we find rational functions $(f_0(t), \ldots, f_n(t))$ such that $P_j(f_0, \ldots, f_n) = 0$, we can think of this as a mapping from $\mathbf{P}^1$ to X. We call this a **rational curve**.
- If we find complex entire functions $(f_0, \ldots, f_n)$ such that $P_j(f_0, \ldots, f_n) = 0$, then we can think of this as a holomorphic map from $\mathbf{C}$ to X, and we call this a **holomorphic curve**.
- If we find non-Archimedean entire functions $(f_0, \ldots, f_n)$ such that $P_j(f_0, \ldots, f_n) = 0$, then we can think of this as an analytic map from $\mathbf{A}^1$ to X, and we call this a **non-Archimedean analytic curve.**

Picard for Hyperelliptic Curves

Let $P(x)$ be a polynomial without repeated zeros. The variety defined by $y^2 = P(x)$ is called a **hyperelliptic curve.**

Proposition

If $P(x)$ is a polynomial without repeated zeros of degree at least 3, then there do not exist non-constant non-Archimedean meromorphic functions g and h such that $g^2 = P(h)$.

Proof.

Consider the hyperelliptic curve C defined by $y^2 = P(x)$ and the differential $\omega = dx/y = 2dy/P'(x)$.
Since y and $P'(x)$ do not simultaneously vanish and since $\deg P \geq 3$, ω has no poles on C (including at points at infinity).
Hence a solution $g^2 = P(h)$ in meromorphic functions g and h thought of as a map f from $\mathbf{A}^1$ to C results in the pull-back $f^*\omega = \zeta(z)dz$, where ζ is analytic. Then,

$$|\zeta|_r = \left|\frac{h'}{g}\right|_r = \left|\frac{h'}{h}\right|_r \left|\frac{h}{g}\right| \leq \frac{1}{r}\frac{|h|_r}{|g|_r}$$

by the logarithmic derivative lemma. On the other hand,

$$|\zeta|_r = \left|\frac{h'}{g}\right|_r = \left|\frac{2g}{P'(h)}\right|_r = |2|\frac{|g|_r}{|h|_r}\frac{|h|_r}{|P'(h)|_r}.$$

When $|h|_r$ is large, we can estimate $|P'(h)|_r$ from below by $|h|_r^2$, and so we see $|\zeta|_r \to 0$ as $r \to \infty$, and so f is constant. □

Picard-type Theorems

Theorem (Berkovich Picard Theorem)

A non-Archimedean analytic map from $\mathbf{A}^1$ to an algebraic curve of genus at least one must be constant.

Theorem (Cherry's Thesis)

A non-Archimedean analytic map from $\mathbf{A}^1$ to an Abelian variety must be constant.

Remark

This was later extended to semi-Abelian varieties in joint work with An and Wang.

Theorem (Cherry/Ru)

A non-Archimedean analytic map in characteristic zero from $\mathbf{A}^1$ to a non-singular variety X of dimension n admiting holomorphic or logarithmic one forms $\omega_1, \ldots, \omega_n$ such that $\omega_1 \wedge \cdots \wedge \omega_n \not\equiv 0$ must be constant.

Cherry Conjecture

Definition

A variety Z is called **rationally connected** *if given any two $x, y \in Z$, x and z can be connected by a finite chain of rational curves.*

Conjecture (Cherry)

Let $f : \mathbf{A}^1 \to X$ be a non-Archimedean analytic map to a non-singular projective variety X. Then, the image of f is contained in a rationally connected subvariety Z of X.

Remark

Perhaps the easiest case where this conjecture is not known is the case where X is a K3-surface. K3-surfaces contain infinitely many rational curves, so there is not a fixed proper subvariety containing the image of all non-constant non-Archimedean analytic curves.

Algebraic degeneracy

Theorem (An/Cherry/Wang)

Let Y be a possibly singular projective variety and let $\iota : Y \to X$ be a morphism to a smooth projective variety X. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible effective divisors on X such that $\{\iota^* D_i\}_{i=1}^{\ell}$ have rank ℓ in the free Abelian group of divisors on Y. Assume that ℓ is larger than the rank of the subgroup generated by the $c_1(D_i)$ in $\mathrm{NS}(X)$. Then, any analytic map from $\mathbf{A}^1$ to Y is either algebraically degenerate or intersects the support of at least one of the $\iota^* D_i$.

Example

Let f be an algebraically non-degenerate analytic map from $\mathbf{A}^1$ to $\mathbf{A}^2$. Let X be obtained by blowing up $r-1$ general points in $\mathbf{P}^2$, none of which are contained in the hyperplane H at infinity and which are also not contained in the image of f. Let $\{D_i\}_1^r$ consist of the $r-1$ exceptional divisors and the strict transform of H. Then, lifting f to X results in an algebraically non-degenerate map omitting r effective divisors.

Theorem (An/Cherry/Wang)

Let $\iota : Y \to X$ and $f : \mathbf{A}^1 \to Y$ be algebraically non-degenerate. Suppose $\{\iota^* D_i\}_{i=1}^{\ell}$ form a subgoup of rank $\ell > \mathrm{rk}\langle c_1(D_i)\rangle$ in the free Abelian group of Cartier divisors on Y. Then, f intersects some D_i.

Proof.

Let $f : \mathbf{A}^1 \to Y$ and lift to the normalization $\widetilde{Y}$.
We can find integers a_i not all zero so that $\sum a_i c_1(D_i) = 0$. Thus, $\sum a_i \tilde{\iota}^* D_i$ is a non-zero divisor algebraically equivalent to zero on $\widetilde{Y}$.
If there is a non-constant rational map from $\widetilde{Y}$ to an Abelian variety, then f is already algebraically degenerate. Hence, assume $\mathrm{Pic}^0(\widetilde{Y})$ is trivial.
Find a non-constant rational function h on $\widetilde{Y}$ such that

$$\mathrm{div}(h) = \sum a_i \tilde{\iota}^* D_i.$$

If f omits the supports of all the $\iota^* D_i$, then $h \circ f$ is an entire function without zeros, and hence constant. □

Adding the assumption that the D_i are ample to the previous result, combining this with the result of Lin and Wang, and then fixing the argument in Noguchi Winkelmann gives:

Corollary

Let Y be a closed positive dimensional subvariety of a non-singular projective variety X. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible, effective, ample divisors in general position on X. Let r be the rank of the subgroup of $\mathrm{NS}(X)$ generated by $\{c_1(D_i)\}_{i=1}^{\ell}$. If there exists an algebraically non-degenerate analytic map from **F** *to Y omitting each of the D_i that does not contain all of Y, then*

$$\ell \leq \min\{r + \mathrm{codim}\, Y, \dim X\}.$$

Remark

- *When $X = \mathbf{P}^n$, the above inequality was proven by An, Wang, and Wong.*
- *With the assumption that the components D_i are ample, I suspect a bound can be independent of r.*